Causal Structure of General Relativistic Spacetimes

A *general relativistic spacetime* is a pair M, g_{ab} where M is a differentiable manifold and g_{ab} is a Lorentz signature metric $(+ + +...-)$ defined on all of M.

(1) Existence of a Lorentzian metric.

What conditions must M satisfy to admit a Lorentz signature metric?

> *Lemma*: A necessary and sufficient condition for M to admit a Lorentzian metric is the existence of a continuous non-vanishing line element field on M, i.e. a continuous assignment to each $p \in M$ of a one-dimensional subspace of M_p.

(i) Any non-compact M will do. (ii) Not any compact M will do. In the case of $\dim(M) = 2$ signature $(+-)$ (a) torus $S^1 x S^1$ will do, but (b) S^2 will not do—can't comb a two-sphere! In general, what topological conditions must a compact n-dimensional M satisfy?

(2) Temporal orientability.

M, g_{ab} is *temporally orientable* iff it admits a globally consistent time sense.

> *Lemma*: The following conditions are equivalent:
>
> (i) There exists a continuous non-vanishing timelike vector field on M.
>
> (ii) Any means of transporting a timelike vector around an arbitrary closed loop in spacetime that is continuous and keeps the vector timelike does not result in time inversion of the vector when it returns to the starting point.
>
> (iii) Parallel transport of a timelike vector around an arbitrary closed loop in the spacetime does not result in time inversion.
>
> *Lemma*: For any M, g_{ab} if M is simply connected, then M, g_{ab} is temporally orientable.
>
> *Cor*: If M, g_{ab} is not temporally orientable, then temporal orientability can be achieved by passing to a covering spacetime.

(3) Chronology.

Suppose that M, g_{ab} is temporally orientable. Choose one of the two possible orientations as giving the future direction of time.

Def.: For $p, q \in M$, $p << q$ (p *chronologically precedes* q) iff there is a future directed timelike curve from p to q.

Lemma: $<<$ is a transitive relation.

Def.: The time oriented spacetime M, g_{ab} has a *time order* iff $<<$ is irreflexive, i.e. $\forall p \in M \ \sim (p << p)$ (i.e. there are no closed future directed timelike curves (CTCs), a.k.a. the *chronology condition*).

Lemma: For any time oriented spacetime M, g_{ab}, if M is compact then there are CTCs.

Def.: A violation of the chronology condition is *intrinsic* to M, g_{ab} if it does not result from making identifications in a larger spacetime. This will be the case if M is simply connected. Gödel spacetime will prove to be such a case.

Def.: The *chronological past* of p is $I^-(p) := \{q \in M : q << p\}$. The *chronological future* $I^+(p)$ of p is defined analogously.

Def.: A spacetime is *chronologically vicious* iff $\forall p \in M \ p << p$.

Def.: A spacetime M, g_{ab} is *reflecting* iff for all $p, q \in M$, $I^+(p) \supseteq I^+(q) \Leftrightarrow I^-(q) \subseteq I^-(q)$.

Lemma: Let M, g_{ab} be a reflecting spacetime. If there is a CTC trough any point of M, g_{ab} then the spacetime is chronologically vicious.

(4) <u>Causality</u>.

Def.: $p < q$ (p causally precedes q) iff there is a future directed causal curve from p to q.

Def.: The *causal past* of p is $C^-(p) := \{q \in M : q < p\}$. The *causal future* $C^+(p)$ of p is defined analogously.

Lemma: $<$ is a transitive relation.

Def.: The time oriented spacetime M, g_{ab} has a *causal order* iff $<$ is irreflexive, i.e. $\forall p \in M \ \sim (p < p)$ (i.e. there are no closed future directed causal curves (CCCs), a.k.a. the basic *causality condition*).

Lemma: The causality condition is stronger than the chronology condition.

(5) <u>Past and future distinguishing</u>.

Def.: A time oriented spacetime M, g_{ab} is *past* (respectively, *future*) *distinguishing* iff $\forall p, q \in M$ $[(I^-(p) = I^-(q)) \Rightarrow p = q]$ (respectively, $[(I^+(p) = I^+(q)) \Rightarrow p = q]$).

Lemma: Past and future distinguishing are stronger than the causality condition. If a spcetime is reflecting, then it is both past and future distinguishing if it is either.

(6) Strong causality.

Def.: A time oriented spacetime spacetime M, g_{ab} is *strongly causal* iff it possesses no almost closed causal curves, i.e. $\forall p \in M$ and any open neighborhood $N(p)$ there is a subneighborhood $N'(p) \subset N(p)$ such that once a future directed causal curve leaves $N'(p)$ it never renters.

Lemma: Strong causality is stronger than past and future distinguishing.

Lemma: If M, g_{ab} is strongly causal then the manifold topology of M coincides with the Alexandrov topology where a basis of open sets is given by sets of the form $I^-(p) \cap I^+(q)$ for $p, q \in M$.

(7) Stable causality.

Def. A time oriented spacetime spacetime M, g_{ab} is *stably causal* iff it satisfies basic causality and there exists a metric g'_{ab} such that at every point $p \in M$ the null cone of g'_{ab} is wider than the null cone of g_{ab} but M, g'_{ab} admits no closed causal curves.

Lemma: Stable causality is stronger than strong causality.

Lemma: Stable causality is the necessary and sufficient condition for the existence of a *global time function*, i.e. a differentiable map $t : M \to \mathbb{R}$ such that whenever $p << q$, $t(p) < t(q)$.

Lemma: For a refletcting spacetime all of the following are equivalent: (i) future distinguishing, (ii) past distinguisging, (iii) stable causality, (iv) strong causality.

(8) Global hyperbolicity.

Def.: A spacetime spacetime M, g_{ab} is *globally hyperbolic* iff it is strongly causal and $\forall p, q \in M$ such that $p << q$, $C^-(q) \cap C^+(p)$ is compact.

Lemma: M, g_{ab} is globally hyperbolic iff it admits a *Cauchy surface*, i.e. a spacelike hypersurface Σ which meets every maximally extended timelike curve exactly once.

Lemma: If Σ is a Cauchy surface for M, g_{ab} then M is topologically and differentiably $\Sigma x \mathbb{R}$, where $\Sigma x\{t\}$ is a Cauchy surface for every $t \in \mathbb{R}$ (no topology change).

(9) Domains of dependence and Cauchy horizons

Def. Let $\Sigma \subset M, g_{ab}$ be an achronal set, i.e. it is not intersected more than once by any future directed timelike curve. The the future domain of dependence $D^+(\Sigma)$ of Σ consists of all points $p \in M$ such that every past intextendible causal curve through p intersects Σ. The *past domain of dependence* $D^-(\Sigma)$ is defined analogously. The *total domain of dependence* $D(\Sigma) := D^+(\Sigma) \cup D^-(\Sigma)$. If $D^+(\Sigma) = M$ then Σ is a Cauchy surface.

Def. The future Caucy horizon $H^+(\Sigma)$ of Σ is the future boundary of $D^+(\Sigma)$ (i.e. $\overline{D^+(\Sigma)} - I^-(D^+(\Sigma))$, where the overbar denotes topological closure.

Lemma: $H^+(\Sigma)$ is achronal and closed. It is generated by null geodesics that are either past intendible in $H^+(\Sigma)$ or else have a past endpoint on the edge of Σ.

(10) Chronal isomorphisms

Def.: Let M, g_{ab} and M', g'_{ab} be two spacetimes. A one-one map $\phi : M \to M'$ is a chronal isomorphism iff for all $p, q \in M$, $p << q \Leftrightarrow \phi(p) << \phi(q)$.

Lemma: In Minkowski spacetime, the chronal isomorphisms are generated by the inhomogeneous Lorentz transformations and dilations.

Lemma: Let $\phi : (M, g_{ab}) \to (M', g'_{ab})$ be a chronal isomorphism. Then the Alexandrov topologies on M and M' are homeomorphic. But in general the manifold topologies need not be the same. But if the spacetimes are past and future distinguishing, ϕ is a homeomorphosm of the manifold topologies (Malament). Further, ϕ is also a smooth conformal isometry so that the causal structures of M, g_{ab} and M', g'_{ab} are the same (Hawking, King, and McCarthy).

M, g_{ab}, T^{ab} is a solution to Einstein's field equations with positive cosmological constant.

(i) The Gödel universe is dust filled, i.e. $T^{ab} = \rho V^a V^b$ where ρ is the density of the dust and V^a is the four-velocity field of the dust. The dust is everywhere rotating, i.e. $\nabla_{[a} V_{b]} \neq 0.$, so the flow lines are not hypersurface orthogonal (by Frobenius' theorem).-+

(ii) The metric is stationary.

(iii) $M = \mathbb{R}^4$. Thus, the spacetime is temporally orientable. Choose a time orientation.

(iv) The spacetime is chronologically vicious. And it is intrinsically so.

(v) There does not exist a single global time slice, i.e. spacelike hypersurface without edges.

(vi) There are no closed time geodesics, so to do time travel you need a rocket. Let $\gamma(\tau)$ be a timelike curve parameterized by proper time τ, and let $a(\tau)$ be the magnitude of the four-acceleration along $\gamma(\tau)$. The total acceleration along $\gamma(\tau)$ is defined as $TA(\gamma) := \int_\gamma a(\tau) d\tau$. It can be shown that

$$TA(\gamma) \leq \ln(m_i/m_f)$$

where m_i and m_f are respectively the initial and final masses of the rocket. The fuel expendede is $m_i - m_f$. Thus the percentage of the initial mass of the rocket is fuel is $\geq 1 - 1/\exp(TA(\gamma))$. It is conjectured that for any CTC in Gödel spacetime, $TA(\gamma) \geq 2\pi(9 + 6\sqrt{5})^{1/2}$. If so, the percentage of the initial mass of the rocket that is fuel differs from 100% by less than one part in $2x10^{-12}$.

(vii) There is no natural sense in which time in the Gödel universe is "circular."